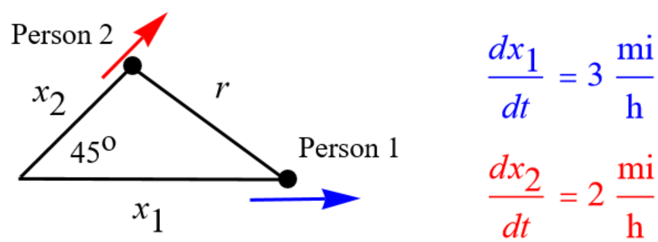
Exercise 48

Two people start from the same point. One walks east at 3 mi/h and the other walks northeast at 2 mi/h. How fast is the distance between the people changing after 15 minutes?

Solution

Draw a schematic of the paths at a certain time.



The aim is to find dr/dt when $t = 15$ min $= (1/4)$ hour. Start with the formula relating the sides of this triangle, the law of cosines.

$$\begin{aligned} r^2 &= x_1^2 + x_2^2 - 2x_1x_2\cos 45^\circ \\ &= x_1^2 + x_2^2 - 2x_1x_2\left(\frac{\sqrt{2}}{2}\right) \\ &= x_1^2 + x_2^2 - \sqrt{2}\,x_1x_2 \\ r &= \sqrt{x_1^2 + x_2^2 - \sqrt{2}\,x_1x_2} \end{aligned}$$

Take the derivative of both sides with respect to time by using the chain and product rules.

$$\begin{aligned} \frac{d}{dt}(r) &= \frac{d}{dt}\left(\sqrt{x_1^2 + x_2^2 - \sqrt{2}\,x_1x_2}\right) \\ \frac{dr}{dt} &= \frac{1}{2}\left(x_1^2 + x_2^2 - \sqrt{2}\,x_1x_2\right)^{-1/2} \cdot \frac{d}{dt}\left(x_1^2 + x_2^2 - \sqrt{2}\,x_1x_2\right) \\ &= \frac{1}{2}\left(x_1^2 + x_2^2 - \sqrt{2}\,x_1x_2\right)^{-1/2} \cdot \left[\frac{d}{dt}(x_1^2) + \frac{d}{dt}(x_2^2) - \sqrt{2}\,\frac{d}{dt}(x_1x_2)\right] \\ &= \frac{1}{2\sqrt{x_1^2 + x_2^2 - \sqrt{2}\,x_1x_2}} \cdot \left[2x_1 \cdot \frac{dx_1}{dt} + 2x_2 \cdot \frac{dx_2}{dt} - \sqrt{2}\left(\frac{dx_1}{dt}x_2 + x_1\frac{dx_2}{dt}\right)\right] \end{aligned}$$

Therefore, when person 1 and person 2 travel the respective distances, $x_1 = 3 * (1/4)$ miles and $x_2 = 2 * (1/4)$ miles, 15 minutes later, the rate of change of the distance between the people with respect to time is

$$\begin{aligned}
\left.\frac{dr}{dt}\right|_{\substack{x_1=3/4\\ x_2=1/2}} &= \frac{1}{2\sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)}}\left\{2\left(\frac{3}{4}\right)(3) + 2\left(\frac{1}{2}\right)(2) - \sqrt{2}\left[(3)\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)(2)\right]\right\} \\
&= \sqrt{13 - 6\sqrt{2}}\,\frac{\text{mi}}{\text{h}} \\
&\approx 2.12479\,\frac{\text{mi}}{\text{h}}.
\end{aligned}$$